KSETA TOPICAL COURSE:

FEYNMAN DIAGRAMS IN CONDENSED MATTER

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FEYNMAN DIAGRAMS IN CONDENSED MATTER

Quantum field theory is applied in all fields of theoretical physics. Many similarities but problems and techniques might also differ. Compare

high-energy physics $\longleftrightarrow$ condensed matter physics

zero temperature $\qquad$ finite temperature

$T=0$ $\qquad$ $T>0$

few-particle problems $\longleftrightarrow$ many-particle problems

e.g. e-e interaction in the presence of Fermi sea

$\rightarrow$ symmetry broken phases like magnetism, superconductivity

Lorentz invariant field theories $\longleftrightarrow$ Lorentz invariance in general broken

REMINDER: GREEN FUNCTION AT T=0

$$\boxed{G(xt, x't') = -i \langle 0 | T\{\psi(xt)\, \psi^\dagger(x't')\} | 0 \rangle}$$

with creation / annihilation operator in the Heisenberg representation ($\hbar = 1$)

$$\psi(xt) = e^{iHt}\, \psi(x)\, e^{-iHt}$$

$$\psi^\dagger(xt) = e^{iHt}\, \psi^\dagger(x)\, e^{-iHt}$$

and time-ordering operator

$$T\{\hat{A}(t)\, \hat{B}(t')\} = \begin{cases} A(t)\, B(t') & \text{if } t > t' \\ -B(t')\, A(t) & \text{if } t < t' \text{ and } A, B \text{ fermionic} \\ B(t')\, A(t) & \text{else} \end{cases}$$

$$G(xt, x't') = -i \langle 0 | \psi(xt)\, \psi^\dagger(x't') | 0 \rangle\, \Theta(t-t')$$

$$-i \xi \langle 0 | \psi^\dagger(x't')\, \psi(xt) | 0 \rangle\, \Theta(t'-t)$$

with $\xi = \begin{cases} -1 & \text{fermions} \\ +1 & \text{bosons} \end{cases}$ $\quad \Theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & x < 0 \end{cases}$

$\hat{=}$ amplitude of finding particle at position x at time t after creating a particle at x' at t'

$$x't' \longrightarrow xt$$

<u>equation of motion:</u>

example: free particle with Hamiltonian

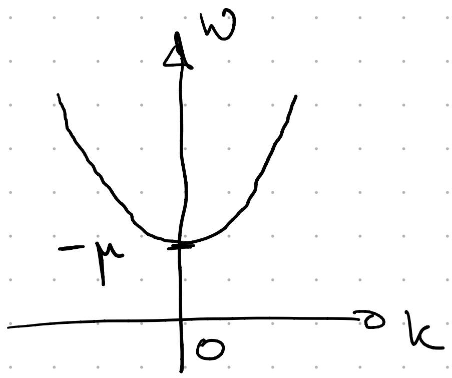
$$\mathcal{H} = \int d^d x \; \psi^\dagger(x) \left(-\frac{\nabla^2}{2m} - \mu \right) \psi(x)$$

μ: chemical potential

using Heisenberg equation of motion $i\partial_t \psi(xt) = [\psi(xt), \mathcal{H}]$ and $\partial_t \Theta(t) = \delta(t)$ one can show [exercise]

$$\boxed{\left(i\partial_t + \frac{\nabla^2}{2m} + \mu \right) G(xt, x't') = \delta(x-x')\,\delta(t-t')}$$

$\to$ spectrum $\omega = \frac{k^2}{2m} - \mu$



solution: Fourier transform

$$G(xt, x't') = \int \frac{d^dk}{(2\pi)^d} \frac{d\omega}{2\pi} \, e^{ik(x-x') - i\omega(t-t')} \, G(k,\omega)$$

$$\rightarrow \quad \boxed{\left(\omega - \frac{k^2}{2m} + \mu\right) G(k,\omega) = 1}$$

solution for G requires regularization of the pole at $\omega = \frac{k^2}{2m} - \mu$

$\rightarrow$ boundary condition for Green function at $t = t'$

$$G(k, t-t') = -i \langle 0 | T\{ \psi_k(t) \psi_k^+(t') \} | 0 \rangle$$

with $\psi_k \psi_{k'}^+ - \xi \, \psi_{k'}^+ \psi_k = \delta_{kk'}$

$$\xi = \left\{ \begin{array}{ll} -1 & \text{fermions} \\ 1 & \text{bosons} \end{array} \right\}$$

$$\Rightarrow \quad \boxed{\begin{aligned} G(k,+0) &= -i \langle 0 | \psi_k \psi_k^+ | 0 \rangle = \\ &= -i \langle 0 | 1 + \xi \psi_k^+ \psi_k | 0 \rangle = -i(1 + \xi n_k) \\ G(k,-0) &= -i\xi \langle 0 | \psi_k^+ \psi_k | 0 \rangle = -i \xi n_k \end{aligned}}$$

with n_k : number of particles with momentum k in the ground state

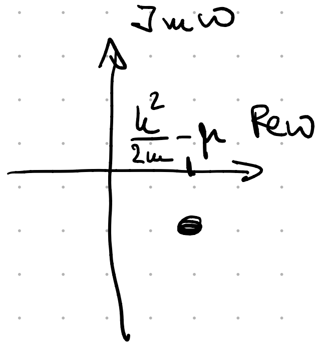
bosons with $\mu < 0$, no Bose-Einstein condensation

$\to$ $|0\rangle$ vacuum empty of particles $\to$ $n_k = 0$

$$\to \int \frac{d\omega}{2\pi} e^{-i\omega t} G(k,\omega) \xrightarrow{t \to 0} \begin{cases} -i & \text{for } t \to 0^+ \\ 0 & \text{for } t \to 0^- \end{cases}$$

single pole in the lower half-plane

$$\boxed{G(k,\omega) = \frac{1}{\omega - \frac{k^2}{2m} + \mu + i0}}$$



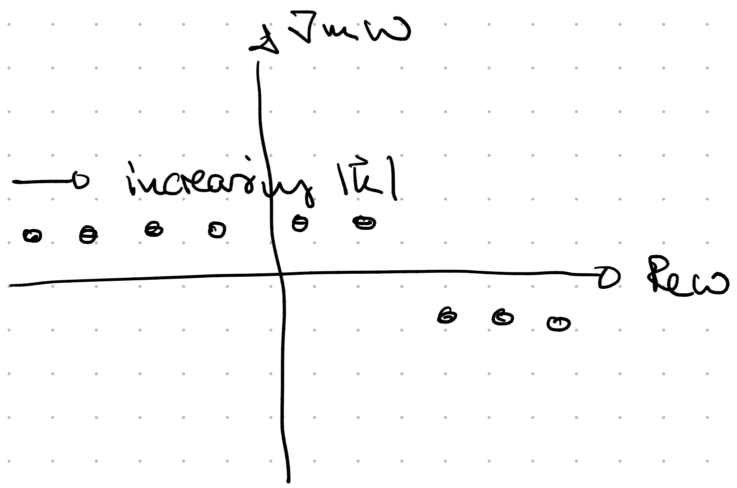
fermions $|0\rangle \hat{=}$ Fermi sea filled up to $|\vec{k}| = k_F = \sqrt{2m\mu}$ for $\mu > 0$

$$\to n_k = \Theta(k_F - |\vec{k}|)$$

$$\int \frac{d\omega}{2\pi} e^{-i\omega t} G(k,\omega) \to \begin{cases} -i\,\Theta(k_F - |\vec{k}|) & \text{for } t \to 0^+ \\ i\,\Theta(k_F - |\vec{k}|) & \text{for } t \to 0^- \end{cases}$$

pole structure depends on the magnitude of $\vec{k}$

$$\boxed{G(\vec{k},\omega) = \frac{1}{\omega - \frac{k^2}{2m} + \mu + i0\,\mathrm{sgn}(|\vec{k}|-k_F)}}$$

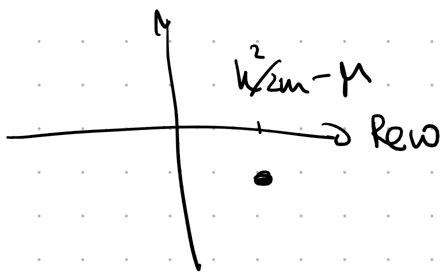


for an empty Fermi sea $k_F = 0$ for $\mu \le 0$ the fermionic Green function coincides with bosonic one.

Retarded Green function

$$\boxed{G^R(k,\omega) = \frac{1}{\omega + i0 - \frac{k^2}{2m} + \mu}}$$

pole in the lower half plane

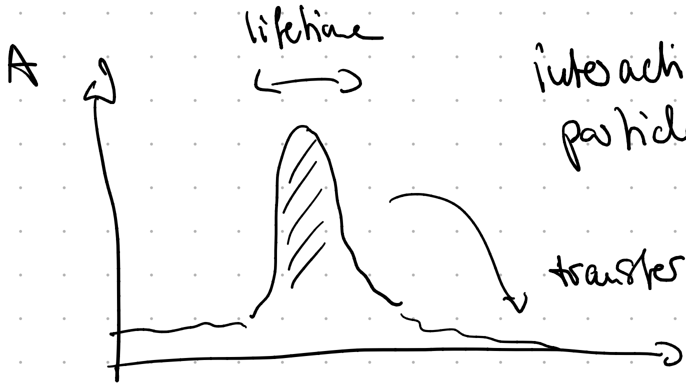


Spectral weight

$$\boxed{A(k,\omega) = -\frac{1}{\pi}\,\mathrm{Im}\,G^R(k,\omega) = \delta\left(\omega - \frac{k^2}{2m} + \mu\right)}$$

free particles
delta function

modifications
due to
interactions



interacting
particles

transfer of spectral weight

APPLICATION: IMPURITY SCATTERING

Hamiltonian $\quad \mathcal{H} = H_0 + V\,\delta(x)$

local impurity at $x=0$

$$H_0 = -\frac{\nabla^2}{2m} - \mu$$

bare retarded Green function

$$G_0^R(k,\omega) = \frac{1}{\omega + i0 - \frac{k^2}{2m} + \mu}$$

spatial Fourier transform

$$G_0^R(x,\omega) = \int \frac{d^dk}{(2\pi)^d}\, e^{ikx}\, G(k,\omega)$$

Full retarded Green function obey

$$\boxed{(\omega - \mathcal{H})\, G^R(x,x';\omega) = \delta(x-x')}$$

Ansatz for full retarded Green function

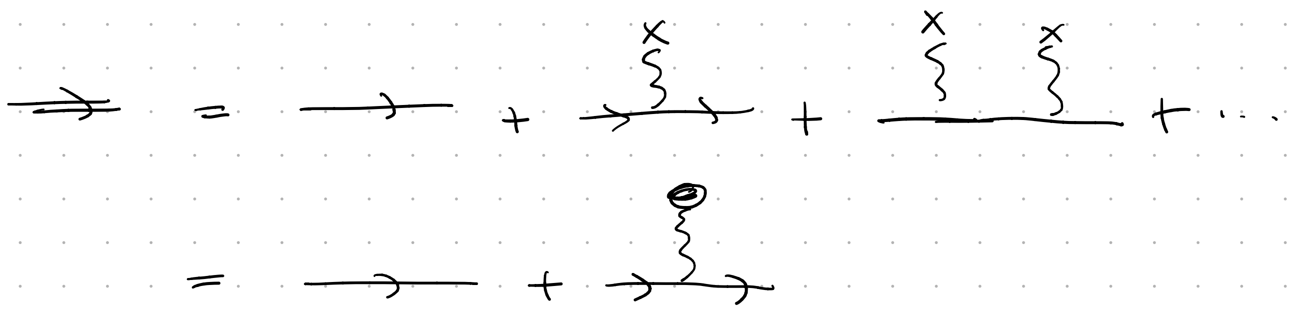
$$G^R(x,x',\omega) = \underbrace{G_0^R(x-x',\omega)}_{\substack{\text{free propagation} \\ \text{without scattering}}} + \underbrace{G_0^R(x,\omega)}_{\substack{\text{propagation} \\ \text{away from} \\ \text{the impurity}}} T(\omega) \underbrace{G_0^R(-x',\omega)}_{\substack{\text{propagation to} \\ \text{the impurity}}}$$

Solution for the T-matrix [exercise]

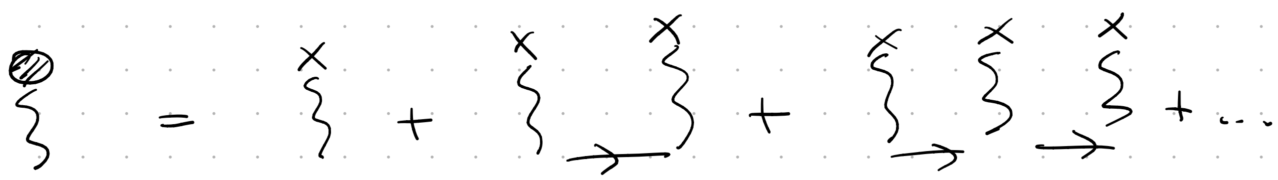
$$\boxed{T(\omega) = \frac{V}{1 - V G_0^R(0,\omega)}}$$

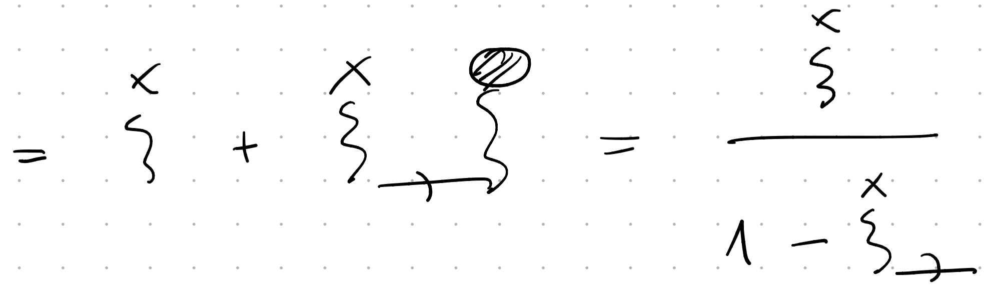
corresponds to a geometric series

interpretation in terms of Feynman diagrams



with T-matrix





FINITE TEMPERATURE QUANTUM FIELD THEORY

The evolution operator can be expressed in terms of a coherent state functional integral

$$U(\phi_f^* t_f, \phi_i, t_i) = \int \mathcal{D}\phi^* \mathcal{D}\phi \, \exp\left[i \int_{t_i}^{t_f} dt \int dx \, \mathcal{L} \right]$$

initial and final states

$\phi(t_i) = \phi_i$

$\phi^*(t_f) = \phi_f^*$

in terms of the Lagrangian

$$\boxed{\mathcal{L} = \phi^* i \partial_t \phi - \mathcal{H}(\phi^*, \phi)}$$

Similarly, the partition function

$$Z = \mathrm{tr}\left\{ e^{-\beta H} \right\}$$

statistical operator $\beta = 1/T$ inverse temperature $(k_B = 1)$

can be expressed in terms of functional integral

$$\boxed{Z = \int \mathcal{D}\phi^* \mathcal{D}\phi \, \exp\left[- \int_0^\beta d\tau \int d^d x \, \mathcal{L} \right]}$$

in terms of the imaginary time Lagrangian

$$\boxed{\mathcal{L} = \phi^* \partial_\tau \phi + \mathcal{H}(\phi^*, \phi)}$$

Wick rotation

$$\boxed{\begin{aligned} i\partial_t &\to \partial_\tau \\ -it &\to \tau \end{aligned}}$$

boundary conditions due to the trace of Z :

$$\phi(\beta) = \zeta \, \phi(0) \quad \text{with} \quad \zeta = \begin{cases} -1 & \text{fermions} \\ 1 & \text{bosons} \end{cases}$$

thermal Green function for free particles

$$-\left(\partial_\tau + \frac{k^2}{2m} - \mu \right) \mathcal{G}(k, \tau) = \delta(\tau)$$

Fourier transform on the finite interval $0 \leq \tau \leq \beta$

$$\boxed{\begin{aligned} \mathcal{G}(k,\tau) &= \frac{1}{\beta} \sum_{\omega_n} e^{-i\omega_n \tau} \, \mathcal{G}(k,\omega_n) \\ \mathcal{G}(k,\omega_n) &= \int_0^\beta d\tau \, e^{i\omega_n \tau} \, \mathcal{G}(k,\tau) \end{aligned}}$$

Matsubara frequencies with $n \in \mathbb{Z}$

$$\omega_n = \begin{cases} (2n+1)\pi T & \text{fermions} \\ 2n\pi T & \text{bosons} \end{cases}$$

$$\Rightarrow \boxed{\mathcal{G}(k,\omega_n) = \frac{1}{i\omega_n - \frac{k^2}{2m} + \mu}}$$

DILUTE WEAKLY INTERACTING BOSE GAS

complex bosonic field ϕ with interaction $u_0 > 0$

action

$$\boxed{S[\phi^*,\phi] = \int_0^\beta d\tau \int d^dx \; \mathcal{L}}$$

$$\mathcal{L} = \phi^*(x\tau)\left[\partial_\tau - \frac{\nabla^2}{2m} - \mu\right]\phi(x,\tau) + \frac{u_0}{2}|\phi(x\tau)|^4$$

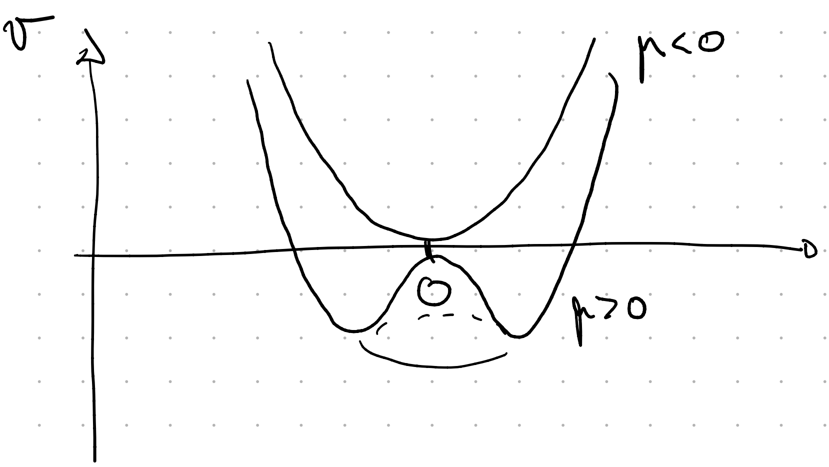
paradigmatic example for $U(1)$ symmetry breaking

consider field configuration homogeneous in time and space

$$\phi(x\tau) \to \phi$$

action reduces to a potential for the mean field

$$\boxed{V(\phi) = -\mu|\phi|^2 + \frac{u_0}{2}|\phi|^4}$$

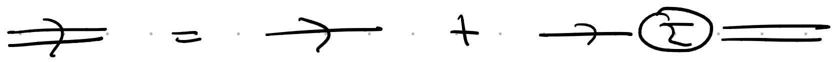


spontaneous symmetry breaking at $\mu = 0$

Self-energy in the non-condensed phase $\mu < 0$

bare Green function $\quad g_0(k\,\omega_n) = \dfrac{1}{i\omega_n - \frac{k^2}{2m} + \mu}$

Dyson equation for the Green function

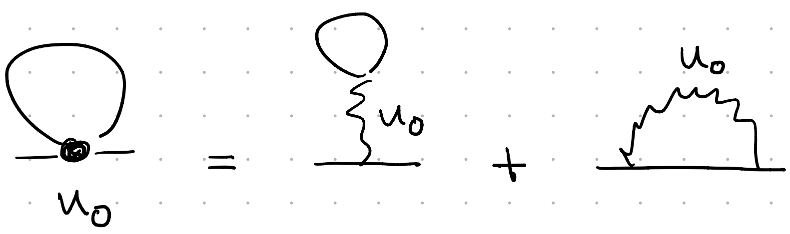


reduces to an algebraic equation in frequency/momentum space

$$g(k\,\omega_n) = g_0(k\,\omega_n) + g_0(k\,\omega_n)\,\Sigma(k\,\omega_n)\,g(k\,\omega_n)$$

$$\Rightarrow \quad \boxed{g^{-1}(k\,\omega_n) = g_0^{-1}(k\,\omega_n) - \Sigma(k\,\omega_n)}$$

one-loop approximation for the self-energy



$$\Sigma_{1L}(k\,\omega_n) = -2u_0\,\frac{1}{\beta}\sum_{k'\omega_n'} g_0(k'\,\omega_n')$$

independent of k and ω_n

explicitly:

$$\Sigma_{1L} = -2u_0 \sum_k \frac{1}{\beta} \sum_{\omega_n} \frac{1}{i\omega_n - \frac{k^2}{2m} + \mu}$$

summation over Matsubara frequencies

$$= 2u_0 \sum_k n_B\left(\frac{k^2}{2m} - \mu\right)$$

[with Bose function $n_B(\varepsilon) = \frac{1}{e^{\beta\varepsilon} - 1}$]

$$= 2u_0 \int_0^\infty \frac{dk}{(2\pi)^d} k^{d-1} S_d \frac{1}{e^{\beta\left(\frac{k^2}{2m} - \mu\right)} - 1}$$

S_d: surface of the d-dimensional unit sphere

for $\mu < 0$: $\Sigma_{1L} = 0$ at $T = 0$ $\leftarrow$ no particles in the ground state

at finite T:

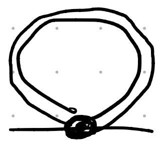
renormalization of the chemical potential

$$\boxed{\mu_{eff}(T) = \mu - \Sigma_{1L} = \mu - u_0 (2mT)^{d/2} \, \Psi(e^{\beta\mu})}$$

where $\Psi(y) = \frac{S_d}{(2\pi)^d} \int_0^\infty dx \, x^{\frac{d-2}{2}} \frac{1}{e^x y^{-1} - 1}$

one-loop approximation for the effective chemical potential can be improved by evaluating Σ_{1L} with the renormalized Green function

$\rightarrow$ self-consistent one-loop approximation



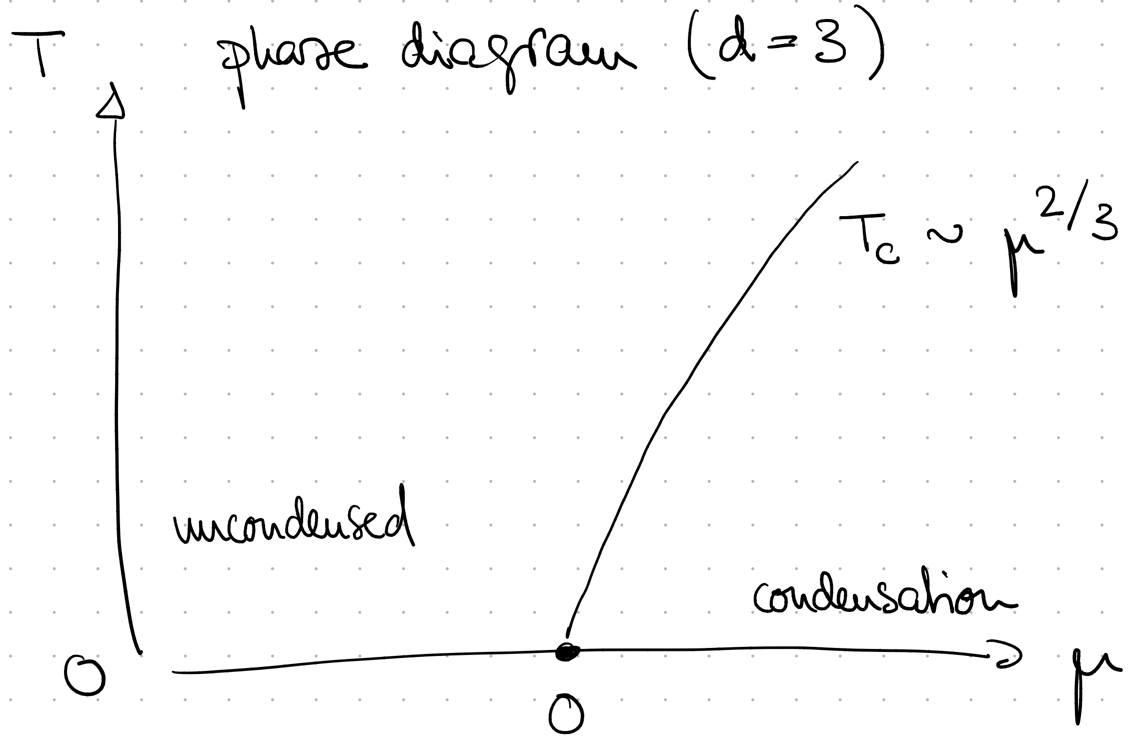
$$\boxed{\mu_{eff}(T) = \mu - u_0 (2mT)^{d/2} \, \Psi\left(e^{\beta \mu_{eff}(T)}\right)}$$

Bose-Einstein condensation for $\mu_{eff}(T_c) \overset{!}{=} 0$

$\rightarrow$ critical temperature $0 = \mu - u_0 (2mT)^{d/2} \, \Psi(1)$

$$\Rightarrow \quad \boxed{T_c \sim \mu^{2/d}}$$

phase diagram $(d=3)$



Vertex corrections (for $\mu<0$)

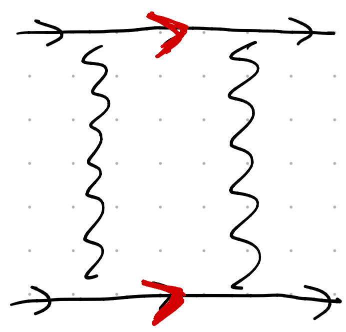
consider renormalization of the interaction amplitude u_0

expand Z up to second order : all connected irreducible contractions that leave two external ϕ^* and two external ϕ legs

$$\left(\frac{Z}{Z_0}\right)^{(2)} = \frac{1}{2!}\left(-\frac{u_0}{2}\right)^2 \int d\tau\, d^dx\; d\tau'\, d^dx'$$

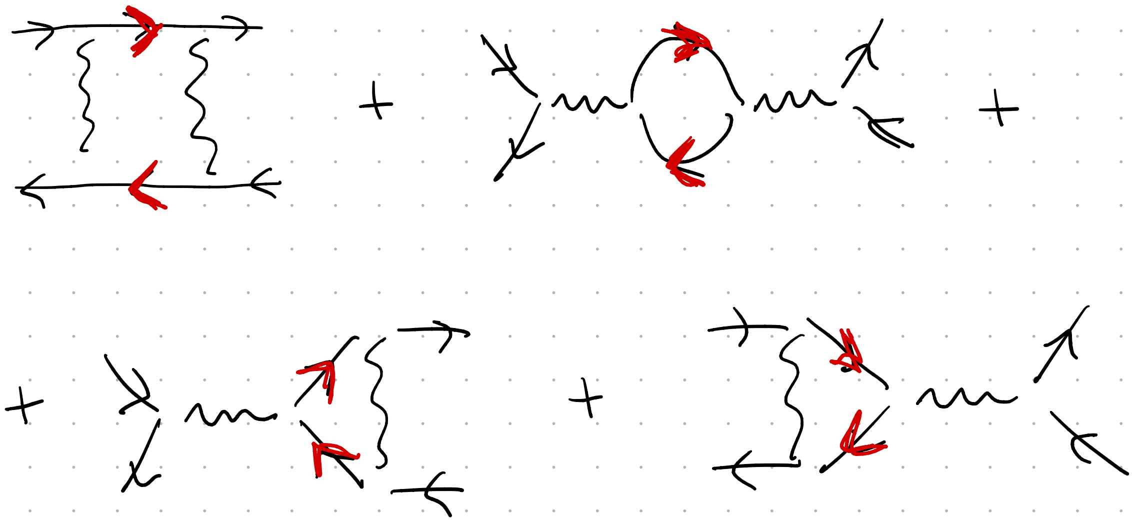
$$\langle \phi^*(x\tau)\phi^*(x\tau)\phi(x\tau)\phi(x\tau)\;\; \phi^*(x'\tau')\phi^*(x'\tau')\phi(x'\tau')\phi(x'\tau') \rangle_0$$

$\times 2$

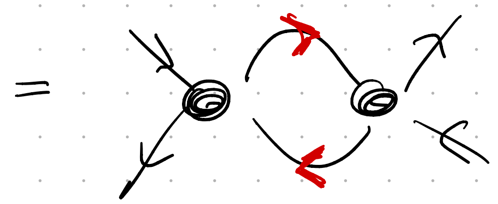


particle - particle channel

+ particle-hole channel



each diagram with combinatorial factor 4



It follows

$$\left(\frac{Z}{Z_0}\right)^{(2)} = \frac{1}{2}\left(-\frac{u_0}{2}\right)^2 \int d\tau d^dx\, d\tau' d^dx' \Big[$$

$$\langle \phi^{*2}(x\tau)\phi^2(x'\tau')\rangle_0 \left(-g_0(x-x',\tau-\tau')\right)^2 4 \qquad \text{pp channel}$$

$$+ \langle \phi^*(x\tau)\phi(x\tau)\ \phi^*(x'\tau')\phi(x'\tau')\rangle_0\ g_0(x-x',\tau-\tau')\, g_0(x'-x,\tau'-\tau)\ 16 \Big]$$

ph channel

$$= -\frac{1}{2}\int d\tau d^dx\, d\tau' d^dx' \Big[\delta\Gamma_{pp}(x-x',\tau-\tau')\, \langle \phi^{*2}(x\tau)\phi^2(x'\tau')\rangle_0$$

$$+ \delta\Gamma_{ph}(x-x',\tau-\tau')\, \langle \phi^*(x\tau)\phi(x\tau)\phi^*(x'\tau')\phi(x'\tau')\rangle_0 \Big]$$

renormalized vertices can be identified by re-exponentiation

<u>vertex in the particle-hole channel</u>

$$\delta\Gamma_{ph}(x-x',\tau-\tau') = -\frac{u_0^2}{4}\, 16\; g(x-x',\tau-\tau')\, g(x'-x,\tau'-\tau)$$

$$\downarrow$$

$$\delta\Gamma_{ph}(p,\Omega_n) = -4u_0^2\,\frac{1}{\beta}\sum_{k\omega_n} g_0(k\omega_n)\, g_0(k-p,\omega_n-\Omega_n)$$

$$= \dots =$$

$$= 4u_0^2 \sum_k \frac{n_B\left(\frac{k^2}{2m}-\mu\right) - n_B\left(\frac{(k-p)^2}{2m}-\mu\right)}{-i\Omega_n - \frac{(k-p)^2-k^2}{2m}}$$

in the dilute limit $|\beta\mu| \gg 1$, $\mu < 0$, $\delta\Gamma_{ph}$ is exponentially small $\sim e^{\beta\mu}$

ADDENDUM:

Main idea for vetex corrections:

$$\langle e^{-\frac{u}{2}x^4}\rangle_0 \cong 1 - \frac{u}{2}\langle x^4\rangle_0 + \frac{1}{2!}\left(-\frac{u}{2}\right)^2 \langle \underbrace{x^4 \; x^4}\rangle_0$$

$$\downarrow$$

$$C\langle x^2 \; g^2 \; x^2\rangle_0$$

$$= 1 - \frac{1}{2}\Big(\underbrace{u - \frac{u^2}{4} C g^2}_{\text{renormalized vetex}}\Big)\langle x^4\rangle_0$$

u_{eff}

$$\cong \langle e^{-\frac{u_{eff}}{2}x^4}\rangle_0$$

vertex in the particle-particle channel

$$\delta\Gamma_{pp}(p,\Omega_n) = -\frac{u_0^2}{4}\, 4\, \frac{1}{\beta}\sum_{k\omega_n} g(k\,\omega_n)\, g(p-k, i\Omega_n - i\omega_n)$$

$$= \ldots = u_0^2 \sum_k \frac{n_B\left(\frac{k^2}{2m}-\mu\right) - n_B\left(-\frac{(p-k)^2}{2m}+\mu\right)}{i\Omega_n - \frac{(p-k)^2+k^2}{2m} + 2\mu}$$

in the dilute limit $|\beta\mu| \gg 1$, $\mu < 0$, this reduces to

$$\delta\Gamma_{pp}(p,\Omega_n) = u_0^2 \sum_k \frac{1}{i\Omega_n - \frac{(k-p/2)^2 + (k+p/2)^2}{2m} + 2\mu}$$

$$= u_0^2 \sum_k \frac{1}{\underbrace{\left(i\Omega_n - \frac{p^2}{4m} + 2\mu\right)}_{\text{energy of centre of mass}} - \underbrace{\frac{k^2}{m}}_{\text{kinetic energy of relative motion}}}$$

$$= u_0^2\, G\left(r=0,\, i\Omega_n - \frac{p^2}{4m} + 2\mu\right)$$

local Green function of relative motion

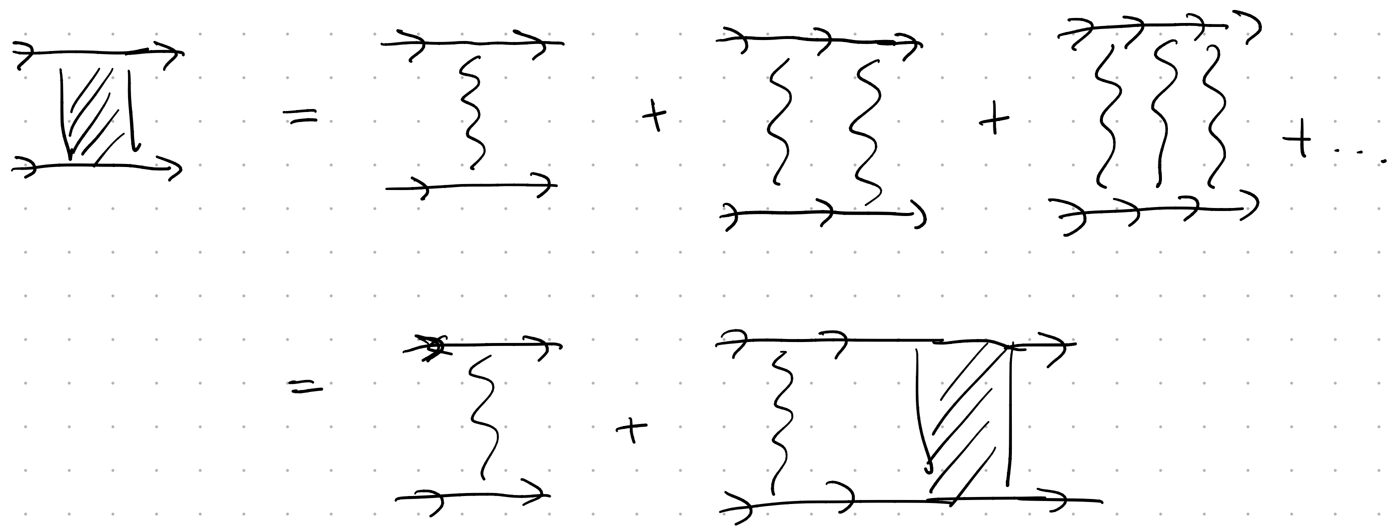
renormalization of the interaction in the dilute limit

$$u_{eff} = u_0 + u_0^2 \, G(0) + O(u_0^3)$$

$\hat{=}$ expansion of the exact two-particle T-matrix up to second order

$$\boxed{T = \frac{u_0}{1 - u_0 G(0)}}$$

$\rightarrow$ in the dilute limit the interaction vertex given by the summation of ladder diagrams in the particle-particle channel



yielding the <u>exact</u> two-particle T-matrix other contributions are exponentially suppressed.

renormalization of the interaction close to quantum criticality: $T=0$ and $\mu \to 0^-$

vertex correction dominated by p-p channel

$$\delta\Gamma_{pp}(0,0) = u_0^2 \sum_k \frac{1}{2\mu - \frac{k^2}{m}} =$$

$$= u_0^2 \frac{S_d}{(2\pi)^d} \int_0^\Lambda dk \frac{k^{d-1}}{2\mu - \frac{k^2}{2m}}$$

$$\xrightarrow{\text{for } \mu \to 0^-} \propto \begin{cases} -\ln \dfrac{\Lambda^2}{2m(-\mu)} & \text{if } d=2 \\ -(-\mu)^{\frac{d-2}{2}} & \text{else} \end{cases}$$

perturbative correction is divergent for $d \leq 2$!

in general for bosons:

perturbation theory near quantum criticality ($T=0$) is controlled as long as the effective dimension $D = d+z > d_c^+$ with the upper

$\boxed{\text{critical dimension } d_c^+ = 4 .}$

where z is the dynamical exponent.

for the critical Bose gas the critical Green function

$$g^{-1}(k, i\omega_n)\Big|_{\mu=0} = i\omega_n - \frac{k^2}{2m} \longrightarrow \omega \sim k^z$$

with $\boxed{z = 2}$

For a Lorentz invariant theory $\omega \sim k$ and $z = 1$. For $d = 3$ Lorentz invariant (massless) theories are at their upper critical dimension and logarithmic divergent in the IR at $T = 0$.

$\Rightarrow$ renormalization group treatment to sum up logarithmic divergencies.

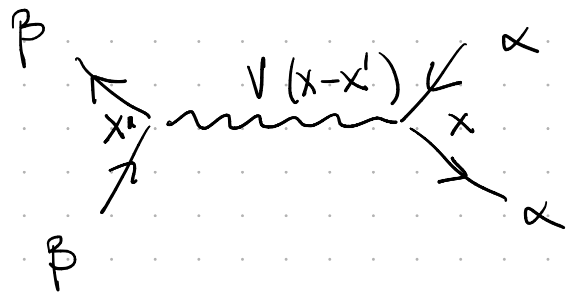
THEORY OF SCREENING

In a Fermi liquid the long-range Coulomb interaction is screened due to vertex corrections.

Consider the electron-electron interaction

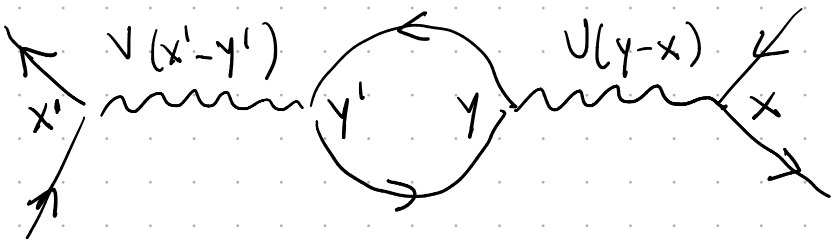
$$S_{int} = \int_0^\beta d\tau \, d^dx \, d^dx' \, \frac{1}{2} V_0(x-x') \sum_{\alpha\beta=\uparrow\downarrow} \psi^+_\alpha(x\tau)\psi^+_\beta(x'\tau)\psi_\beta(x'\tau)\psi_\alpha(x\tau)$$

with the Coulomb interaction $V_0(x-x') = \frac{e^2}{|\vec{x}-\vec{x}'|}$



Fourier transform $V_0(q) = \frac{4\pi e^2}{q^2}$

The vertex correction diagram of the particle-hole channel



directly leads to a renormalization of the Coulomb interaction vertex

$$\Gamma(x-x',\tau-\tau') = V_0(x-x')\,\delta(\tau-\tau') + \underbrace{\delta\Gamma(x-x',\tau-\tau')}_{\text{retardation}}$$

$$\delta\Gamma(x-x',\tau-\tau') = -\int d^dy\, d^dy'\, \frac{V(y-x)V(x'-y')}{4}\, \underbrace{4}_{\text{combinatorics}}\, \underbrace{(-1)}_{\text{fermion loop}}\, \underbrace{2}_{\text{sum over spins}}$$

$$\times \left(-\mathcal{G}_0(y-y',\tau-\tau')\right)\left(-\mathcal{G}_0(y'-y,\tau'-\tau)\right)$$

Fourier transform

$$\delta\Gamma(q,i\Omega_n) = 2\, V^2(q)\, \frac{1}{\beta}\sum_{k\omega_n} \mathcal{G}_0(k,\omega_n)\, \mathcal{G}_0(k-q,\omega_n-\Omega_n)$$

$$= V^2(q)\, \Pi(q,i\Omega_n)$$

with the polarization $\Pi \;\hat{=}$ 

$$\boxed{\Pi(q,i\Omega_n) = 2\,\frac{1}{\beta}\sum_{k\omega_n} \frac{1}{i\omega_n - \frac{k^2}{2m} + \mu}\, \frac{1}{i\omega_n - i\Omega_n - \frac{(k-q)^2}{2m} + \mu}}$$

$$= 2\frac{1}{\beta}\sum_{k\omega_n}\left(\frac{1}{i\omega_n-\frac{k^2}{2m}+\mu}-\frac{1}{i\omega_n-i\Omega_n-\frac{(k-q)^2}{2m}+\mu}\right)\frac{1}{-i\Omega_n-\frac{(k-q)^2-k^2}{2m}}$$

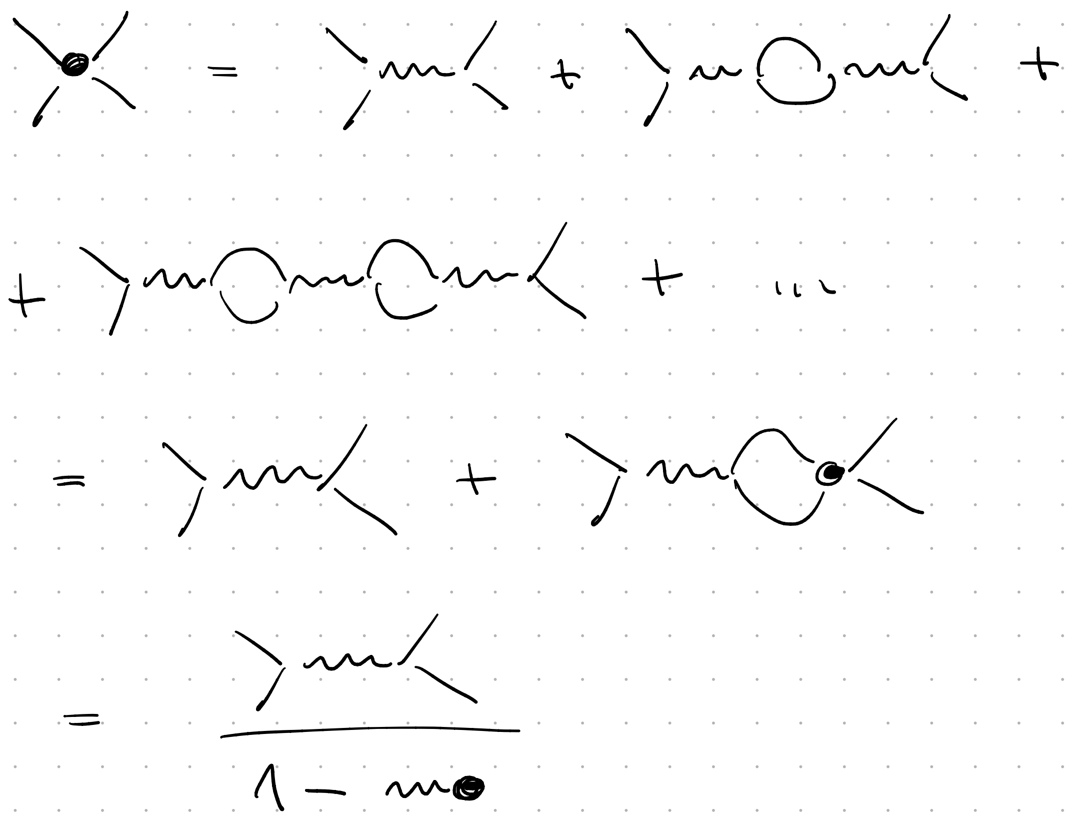
$$= -2\sum_k \frac{f\left(\frac{k^2}{2m}-\mu\right)-f\left(\frac{(k-q)^2}{2m}+\mu\right)}{i\Omega_n+\frac{(q-k)^2-k^2}{2m}}$$

with Fermi function $f(x)=\dfrac{1}{e^{\beta x}+1}$

Effective vertex

$$\boxed{\Gamma(q,\Omega_n) = V_0(q) + V_0^2(q)\,\Pi(q,\Omega_n) + \ldots}$$

summation of the geometric series



$$\Rightarrow \quad \boxed{\Gamma(q,\Omega_n) = \frac{V_0(q)}{1 - V_0(q)\,\Pi(q,\Omega_n)}}$$

random phase approximation (RPA)

Consider static limit $\Omega = 0$ and $\vec{q} \to 0$

$$\Pi(q,0) = -2 \sum_k \frac{f\left(\frac{k^2}{2m} - \mu\right) - f\left(\frac{(k-q)^2}{2m} + \mu\right)}{\frac{(q-k)^2 - k^2}{2m}}$$

$$\xrightarrow{\vec{q} \to 0} \quad 2 \sum_k f'\left(\frac{k^2}{2m} - \mu\right) \overset{T=0}{=} -\nu$$

total density of states at the Fermi level

$$\Rightarrow \quad \boxed{\Gamma(\vec{q},0) \cong \frac{V_0(q)}{1 + \nu V_0(q)} = \frac{4\pi e^2}{q^2 + \nu 4\pi e^2} = \frac{4\pi e^2}{q^2 + k_{TF}^2}}$$

Thomas-Fermi approximation

Coulomb potential transforms into a Yukawa potential with Thomas-Fermi wavevector $k_{TF}^2 = 4\pi e^2 \nu$

In real space:

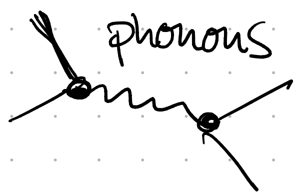
$$\boxed{\Gamma(\vec{x}) = \int \frac{d\vec{q}}{(2\pi)^3} e^{i\vec{q}\vec{x}} \Gamma(\vec{q}) = \frac{e^2}{|\vec{x}|} e^{-k_{TF}|\vec{x}|}}$$

screened Coulomb interaction decays exponentially on the length scale of $1/k_{TF}$.

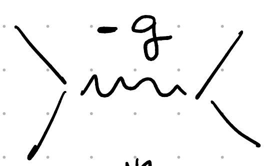
THE COOPER INSTABILITY

phonons dress the e-e interaction further. They mediate an effective e-e interaction that becomes even attractive for small frequencies

- overscreening



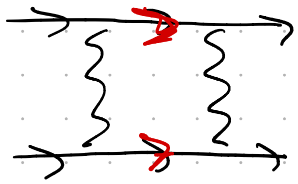
→ consider an effective attractive interaction between electrons for frequencies $|\omega| < \omega_D$ smaller than the Debye frequency



with $g > 0$.

Vertex correction in the particle-particle channel

$$\delta\Gamma_{pp}(q, \Omega_n) = $$



$$= -g^2 \frac{1}{\beta} \sum_{k\omega_n} \mathcal{G}_0(k, \omega_n)\, \mathcal{G}_0(q-k, \Omega_n - \omega_n)$$

$$= \ldots$$

$$= -g^2 \sum_k \frac{f\left(\frac{k^2}{2m} - \mu\right) - f\left(-\frac{(q-k)^2}{2m} + \mu\right)}{i\Omega_n - \left(\frac{(q-k)^2 + k^2}{2m} - 2\mu\right)}$$

Consider the static limit $\Omega = 0$ and $q = 0$, and

abbreviate $\xi_k = \frac{k^2}{2m} - \mu$

$$\delta\Gamma_{pp}(0,0) = -g^2 \sum_k \frac{f(\xi_k) - f(-\xi_k)}{-2\xi_k}$$

$$= g^2 \int d\xi \underbrace{\sum_k \delta(\xi - \xi_k)}_{\nu(\xi) \approx \nu_0} \frac{f(\xi) - f(-\xi)}{2\xi}$$

density of states per spin at the Fermi level

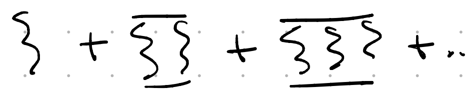
smooth energy dependence

$$= g^2 \nu_0 \int_{-\omega_D}^{\omega_D} d\xi \, \frac{\tanh \frac{\xi}{2T}}{2\xi}$$

Integral diverges logarithmically for $T \to 0$!

$$\boxed{\delta\Gamma_{pp}(0,0) \simeq g^2 \nu_0 \, 2 \int_{T}^{\omega_D} d\xi \, \frac{1}{2\xi} = - g^2 \nu_0 \ln \frac{\omega_D}{T}}$$

Cooper logarithm

summation over geometric series 

$$\Gamma_{pp}(0,0) = \frac{-g}{1 - g\nu_0 \ln \frac{\omega_D}{T}}$$

vertex diverges at the critical temperature

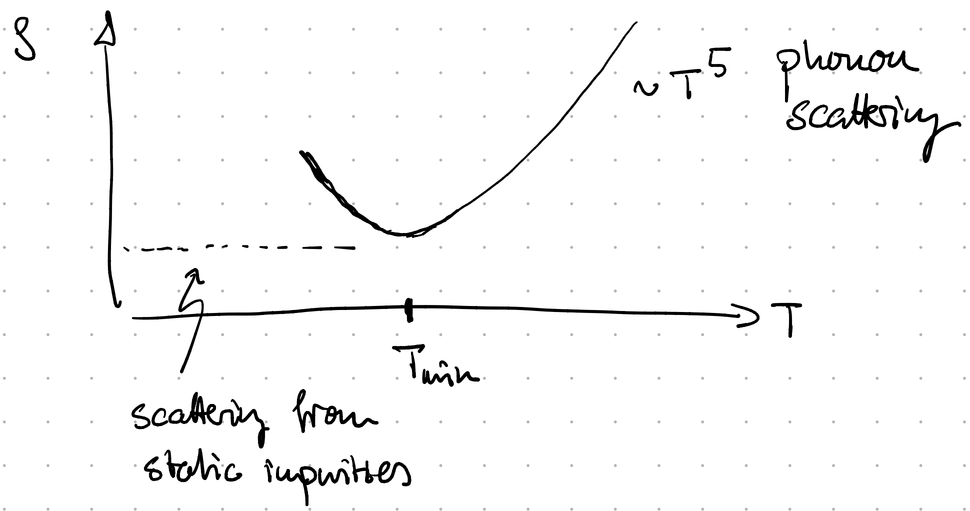
$$\boxed{T_c \sim \omega_D \exp\left[- \frac{1}{g\nu_0} \right]}$$

$\Rightarrow$ superconducting instability

THE KONDO EFFECT

History :

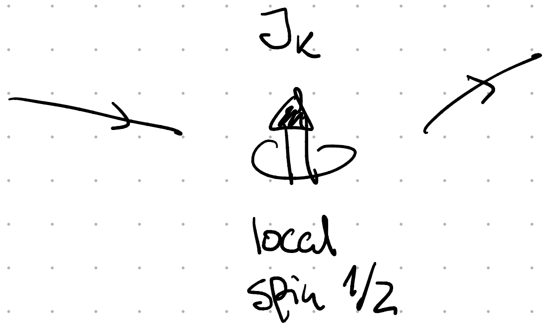
1934 deHaan, deBoer, van den Berg (exp)



minimum in the resistivity of gold $\rho(T)$

1964 Kondo (theory)

scattering of conduction electrons from a magnetic impurity



evaluation of $\delta\rho$ due to magnetic impurities in third order perturbation theory in J_K

$\longrightarrow$ explanation of $T_{min} \propto (n_{imp})^{1/5}$

impurity concentration n_{imp}.

BUT: perturbative expansion divergent for low temperatures

→ characteristic Kondo temperature T_K

"the Kondo problem"

1970 Anderson : one-loop renormalization group

1974 Wilson : numerical renormalization group (NRG)

1975 Nozière : interpretation of strong-coupling fixed point → local Fermi liquid theory

1980 Wiegmann Andrei : Bethe-Ansatz

paradigmatic model for a non-perturbative correlated many-body problem

The Kondo Hamiltonian

$$\boxed{H = \underbrace{\sum_{k\sigma} \xi_k c^+_{k\sigma} c_{k\sigma}}_{\text{Fermi sea of electrons}} + J \vec{s}\vec{S}}$$

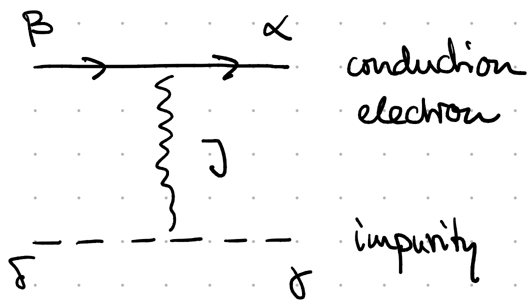
$$\boxed{\text{with } \quad \vec{s} = \sum_{kk'} c^+_{k\alpha} \frac{\vec{\sigma}_{\alpha\beta}}{2} c_{k\beta}}$$

$\vec{S}$: spin operator of the magnetic impurity at $\vec{x} = 0$.

<u>Perturbation theory for the Kondo model</u>

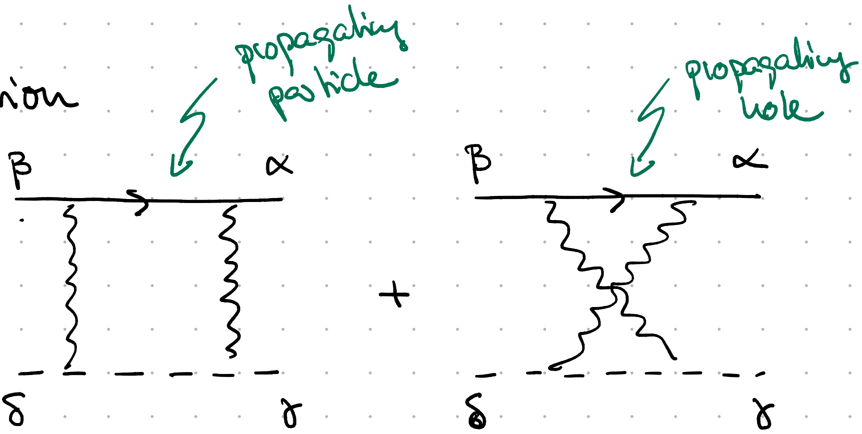
pictorially : the Kondo vertex

$$\Gamma^{(0)}_{\alpha\beta\gamma\delta} = J \frac{\vec{\sigma}_{\alpha\beta}}{2} \vec{S}_{\gamma\delta} =$$



lowest order correction

$$\delta\Gamma_{\alpha\beta\gamma\delta}(\omega) =$$



$$= -\frac{J^2}{2} (S^i S^j)_{\gamma\delta} \left[\sum_k \frac{1-n_k}{\xi_k - \omega} \frac{(\sigma^i \sigma^j)_{\alpha\beta}}{4} + \sum_k \frac{-n_k}{\xi_k - \omega} \frac{(\sigma^j \sigma^i)_{\alpha\beta}}{4} \right]$$

$$= -\frac{J^2}{2} (S^i S^j)_{\gamma\delta} \left[\nu \int_0^\Lambda d\xi \, \frac{1}{\xi - \omega} \frac{(\sigma^i \sigma^j)_{\alpha\beta}}{4} - \nu \int_{-\Lambda}^0 d\xi \, \frac{1}{\xi - \omega} \frac{(\sigma^j \sigma^i)_{\alpha\beta}}{4} \right]$$

with cutoff Λ and DOS ν

$$\cong -\frac{J^2}{2}(S^i S^j)_{\gamma\delta}\, \nu \ln\frac{\Lambda}{\omega} \Bigg(\underbrace{\frac{(\sigma^i\sigma^j)_{\alpha\beta}}{4} - \frac{(\sigma^j\sigma^i)_{\alpha\beta}}{4}}_{= \frac{1}{4} i\varepsilon^{ijk}\sigma^k_{\alpha\beta} \times 2} \Bigg)$$

$$= \dots = J^2 \nu \ln\frac{\Lambda}{\omega}\, \vec{S}_{\gamma\delta}\, \frac{\vec{\sigma}_{\alpha\beta}}{2}$$

$\Rightarrow$ effective Kondo coupling

$$\boxed{J(\omega) = J + J^2\nu \ln\frac{\Lambda}{\omega}}$$

increases for $\omega \to 0$

breakdown of perturbation theory for $J\nu \ln\frac{\Lambda}{\omega_K} \sim 1$

$\Rightarrow$ Kondo temperature $\boxed{T_K \sim \Lambda \exp\left[-\frac{1}{J\nu}\right]}$